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MAXIMALITY OF ANALYTIC OPERATOR ALGEBRAS

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ABSTRACT

Let M be a σ-finite von Neumann algebra and α be an action of $\mathbf{R}$ on M. Let $H^\infty(\alpha)$ be the associated analytic subalgebra; i.e. $H^\infty(\alpha)=\{X\in M: \mathrm{sp}_\alpha(X)\subseteq[0,\infty)\}$. We prove that every σ-weakly closed subalgebra of M that contains $H^\infty(\alpha)$ is $H^\infty(\gamma)$ for some action γ of $\mathbf{R}$ on M. Also we show that (assuming $Z(M)\cap M^\alpha=\mathbf{C}I$) $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M if and only if either $H^\infty(\alpha)=\{A\in M:(I-F)xF=0\}$ for some projection $F\in M$, or $\mathrm{sp}(\alpha)=\Gamma(\alpha)$.

1. Introduction

Let M be a σ-finite von Neumann algebra and let $\alpha=\{\alpha_t: t\in\mathbf{R}\}$ be a continuous action of $\mathbf{R}$ on M, i.e. $\{\alpha_t\}$ is a one-parameter group of $*$-automorphisms of M such that, for each $x\in M$, $t\mapsto\alpha_t(x)$ is σ-weakly continuous. Write

$$H^\infty(\alpha)=\{x\in M:\mathrm{sp}_\alpha(x)\subseteq[0,\infty)\}$$

where $\mathrm{sp}_\alpha(\circ)$ is Arveson's spectrum. The structure of $H^\infty(\alpha)$ was studied by several authors starting with Loebl and Muhly [4] and Kawamura and Tomiyama [2].

It is known that $H^\infty(\alpha)$ can also be defined as the set of all $x\in M$ such that, for every $\rho\in M_*$, the function $t\mapsto\rho(\alpha_t(x))$ lies in the classical Hardy space $H^\infty(\mathbf{R})$. In Theorem 3.15 of [4] it is proved that $H^\infty(\alpha)$ is a σ-weakly closed subalgebra of M containing the identity operator, such that $H^\infty(\alpha)+H^\infty(\alpha)^*$ is σ-weakly dense in M and such that

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$$H^\infty(\alpha) \cap H^\infty(\alpha)^* = M^\alpha \quad (= \{x \in M : \alpha_t(x) = x, t \in \mathbf{R}\}).$$

If $M = L^\infty(\mathbf{R})$ and α_t is a "translation by t" (i.e. $\alpha_t(\theta)(s) = \theta(s-t)$, $\theta \in L^\infty(\mathbf{R})$ $s, t \in \mathbf{R}$) then $H^\infty(\alpha)$ is $H^\infty(\mathbf{R})$. In this case it is well known that $H^\infty(\mathbf{R})$ is a maximal w*-closed subalgebra of $L^\infty(\mathbf{R})$.

As $H^\infty(\alpha)$ can be viewed as a generalization, to a noncommutative setting, of $H^\infty(\mathbf{R})$, it is natural to ask when is $H^\infty(\alpha)$ maximal among the σ-weakly closed subalgebras of M.

In the case when M is commutative it was shown in [9, Corollary 3.1] that $H^\infty(\alpha)$ is maximal if and only if $M^\alpha = \mathbf{C}$. Suppose N is a σ-finite von Neumann algebra and β is a *-automorphism of N preserving a faithful normal state. Let M be the crossed product determined by N and β and let α be the dual (periodic) action. Then it was shown by McAsey, Muhly and Saito in [6–8] that $H^\infty(\alpha)$ is maximal if and only if $M^\alpha (= N)$ is a factor. This result was extended by the author in [14] to show that, whenever α is a periodic action of $\mathbf{R}$ on M and $Z(M) \cap M^\alpha = \mathbf{C}I$ (where $Z(M)$ is the center of M), then $H^\infty(\alpha)$ is maximal if and only if either M^α is a factor or there is a projection $F \in M$ such that $H^\infty(\alpha) = \{x \in M : (I-F)xF = 0\}$. (This is not precisely the way the result is stated in [14] but it can be shown to be equivalent to it.) Finally, it was shown in [10] by Muhly and Saito that in the case of a crossed product by an $\mathbf{R}$-action (with α being the dual action), $H^\infty(\alpha)$ is maximal if and only if M^α is a factor.

In the present paper we settle the general case. We prove (Theorem 3.7) the following:

THEOREM. *Suppose $Z(M) \cap M^\alpha = \mathbf{C}I$. $H^\infty(\alpha)$ is maximal if and only if either* $\mathrm{sp}(\alpha) = \Gamma(\alpha)$ *(where $\Gamma(\alpha)$ denotes the Connes spectrum of α) or there is a projection $F \in M$ such that $H^\infty(\alpha) = \{x \in M : (I-F)xF = 0\}$.*

Note that when α is periodic and $Z(M) \cap M^\alpha = \mathbf{C}I$, $\mathrm{sp}(\alpha) = \Gamma(\alpha)$ if and only if M^α is a factor ([15, 16.4]) and the same holds for crossed products ([15, 21.6]). In general, however, α might satisfy $\Gamma(\alpha) = \mathrm{sp}(\alpha)$ but M^α would not be a factor.

Note also that whenever α is an action of $\mathbf{R}$ on a σ-finite von Neumann algebra M then it is possible to represent M as a direct integral of algebras $\{M(x) : x \in X\}$ in such a way that α induces an action $\alpha(x)$ of $\mathbf{R}$ on $M(x)$ and $Z(M(x)) \cap M(x)^{\alpha(x)} = \mathbf{C}I$ for almost all $x \in X$ (cf. [16, Theorem 8.23]).

When $H^\infty(\alpha)$ is not maximal it was shown, in some cases, that the σ-weakly closed subalgebras of M that contain $H^\infty(\alpha)$ have special properties. For example, it was shown in [14] that, when α is periodic, every such algebra is

$H^\infty(\gamma)$ for some flow γ (also periodic). It was also shown, in this case, that there is a correspondence (one-to-one, under some mild condition) between such algebras and projections of $Z(M^\alpha)$. (This correspondence is described explicitly in Theorem 3.6 of [14].) Related results were obtained in [10] and [13].

When the action α is inner than every σ-weakly closed subalgebra of M that contains $H^\infty(\alpha)$ is $H^\infty(\gamma)$ for some (inner) action γ of $\mathbf{R}$ on M. This was proved by Larson and the author in [3]. In view of this result and the result in the periodic case it was natural to expect that, in general, every σ-weakly closed subalgebra of M containing $H^\infty(\alpha)$ is also an analytic subalgebra (i.e. of the form $H^\infty(\gamma)$ for an action γ of $\mathbf{R}$ on M). In [11] this result was proved by Muhly, Saito and the author in the case where M^α is a Cartan subalgebra of M. In the present paper we prove this result in the general case. In fact we show the following (Theorem 2.19).

THEOREM. *If B is a σ-weakly closed subalgebra of M that contains $H^\infty(\alpha)$ then there is an action γ of $\mathbf{R}$ on M satisfying $H^\infty(\gamma)=B$. In fact, there is a projection $F\in Z(M)\cap M^\alpha$ and a one parameter unitary group $\{v_t : t\in\mathbf{R}\}$, in the center of M^α, such that, for $t\in\mathbf{R}$,*

$$\gamma_t(x)=\begin{cases} x & \text{if } x\in MF,\\ v_t^*\alpha_t(x)v_t & \text{if } x\in M(I-F).\end{cases}$$

2. Algebras containing $H^\infty(\alpha)$

Let M be a σ-finite von Neumann algebra and let $\alpha=\{\alpha_t : t\in\mathbf{R}\}$ be a continuous action of $\mathbf{R}$ on M (i.e. $\alpha_{t+s}=\alpha_t\alpha_s$, $\alpha_{-t}=\alpha_t^{-1}$and, for every $a\in M$, $t\mapsto\alpha_t(a)$ is σ-weakly continuous). The *analytic subalgebra* that is associated with α is

$$H^\infty(\alpha)=\{a\in M : \operatorname{sp}_\alpha(a)\subseteq[0,\infty)\}$$

where $\operatorname{sp}_\alpha(\cdot)$ is Arveson's spectrum. We shall prove (Theorem 2.19) that every σ-weakly closed subalgebra B of M that contains $H^\infty(\alpha)$ is $H^\infty(\gamma)$ for some continuous action γ of $\mathbf{R}$ on M. In fact, there is a projection F in $Z(M)\cap M^\alpha$ (where M^α is the fixed point algebra of α and $Z(M)$ is the center of M) and a one parameter unitary group $\{v_t : t\in\mathbf{R}\}$ in the center of M^α such that $\gamma_t(x)=x$ for $t\in\mathbf{R}$ and $x\in MF$ and $\gamma_t(x)=v_t^*\alpha_t(x)v_t$ for $t\in\mathbf{R}$ and $x\in M(I-F)$.

For a subset S of $\mathbf{R}$ we write $M^\alpha(S)=\{a\in M : \operatorname{sp}_\alpha(a)\subseteq S\}$. We write $B(H)$ for the algebra of all bounded linear operators on a Hilbert space H. For a subset $Y \subseteq H$, $[Y]$ will denote the closed linear subspace spanned by Y. If C is a subalgebra of $B(H)$ and L is a lattice of projections in $B(H)$ then we write

$$\operatorname{alg} L = \{T \in B(H) : (I - P)TP = 0 \text{ for all } P \in L\},$$

$$\operatorname{lat} C = \{P : P \text{ is a projection in } B(H) \text{ and } (I - P)TP = 0 \text{ for all } T \in C\}.$$

By choosing an appropriate representation for M we shall assume, throughout this section, that M has a cyclic and separating vector and we write H for the Hilbert space on which M acts.

We can now use Corollary 3.7 of [5] to conclude that, for every σ-weakly closed subalgebra B of M, $B = \operatorname{alg} \operatorname{lat} B$. We shall fix now a σ-weakly closed subalgebra B of M that contains $H^\infty(\alpha)$.

Let P be a projection in $\operatorname{lat} B \subseteq \operatorname{lat} H^\infty(\alpha)$. Then, as in the proof of [4, Theorem 5.2], let $\tilde{F}_t$, $t \in \mathbf{R}$, be the projection onto $\bigcap_{s<t} [M^\alpha[s, \infty)P(H)]$. Write

$$E_1 = \wedge\{\tilde{F}_t : t \in \mathbf{R}\} \quad \text{and} \quad E_2 = \vee(\tilde{F}_t : t \in \mathbf{R}\}.$$

Then E_1 and E_2 are projections in M'. Write $E = E_2 - E_1$. By construction we have $\tilde{F}_s \leqq \tilde{F}_t$ when $t < s$ and $\tilde{F}_s = \wedge\{\tilde{F}_t : t < s\}$. Hence there is a spectral measure $F(\cdot)$ with values in the projections on $E(H)$ such that $F([t, \infty)) = \tilde{F}_t - E_1$ for $t \in \mathbf{R}$. We define the strongly continuous unitary group $U = \{U_t : t \in \mathbf{R}\}$ on $E(H)$ by

$$U_t = -\int_{-\infty}^{\infty} e^{its} dF(s), \qquad t \in \mathbf{R}.$$

Write K for $E(H)$. We now view M as an algebra of operators on K. For every t, s in $\mathbf{R}$ we have

$$M^\alpha[t, \infty)(\tilde{F}_s - E_1)(K) \subseteq (\tilde{F}_{s+t} - E_1)(K),$$

i.e. $M^\alpha[t, \infty) \subseteq B(K)^\beta[t, \infty)$ where β is the action on $B(K)$ implemented by $U = \{U_t : t \in \mathbf{R}\}$. Using [4, Corollary 2.11] we find that, for $x \in M$, $t \in \mathbf{R}$,

$$\alpha_t(x) = \beta_t(x) = U_t x U_t^*.$$

When M is viewed as acting on H we have

$$\alpha_t(x)E = U_t x E U_t^*, \qquad x \in M, \quad t \in \mathbf{R}.$$

Now note that, for $s < 0$, $[M^\alpha[s, \infty)P(H)] \supseteq P(H)$ and, for $s \geqq 0$, $[M^\alpha[s, \infty)P(H)] \subseteq P(H)$ (as $P \in \operatorname{lat} H^\infty(\alpha)$). Hence $\vee\{\tilde{F}_s : s > 0\} \leqq P \leqq \tilde{F}_0$ and, in particular, P commutes with $\{\tilde{F}_t : t \in \mathbf{R}\}$ and, thus, with $\{U_s : s \in \mathbf{R}\}$.

Suppose now that a lies in $M \cap \mathrm{alg}\{P\}$. Then a lies in $M \cap \mathrm{alg}\{P - E_1\}$ and, for $t \in \mathbf{R}$,

$$\begin{aligned}\alpha_t(a)(P-E_1) &= \alpha_t(a)E(P-E_1) = U_t a E U_t^*(P-E_1) = U_t a E(P-E_1)U_t^* \\ &= U_t(P-E_1)aE(P-E_1)U_t^* = (P-E_1)U_t aE(P-E_1)U_t^* \\ &= (P-E_1)\alpha_t(a)(P-E_1).\end{aligned}$$

Hence $\alpha_t(M \cap \mathrm{alg}\{P\}) = M \cap \mathrm{alg}\{P\}$ for every $t \in \mathbf{R}$ and $P \in \mathrm{lat}\, B$. Since $B = \mathrm{alg\,lat}\, B$, $\alpha_t(B) = B$, $t \in \mathbf{R}$. We therefore have the following.

Proposition 2.1. *Every σ-weakly closed subalgebra B of M that contains $H^\infty(\alpha)$ is α-invariant.*

We write $A = B \cap B^*$. Then $A \supseteq M^\alpha$ and A is α-invariant. For a subset $S \subseteq \mathbf{R}$ we write $A^\alpha(S)$ for $A \cap M^\alpha(S)$. For an element $y \in M$ we write $\mathrm{rp}(y)$ for its range projection. Clearly $\mathrm{rp}(y)$ is in M and if $y \in A$, $\mathrm{rp}(y)$ would lie in A.

Definition 2.2. For $t \in \mathbf{R}$ we define

$$f_t = I - \sup\{\mathrm{rp}(y) : y \in A^\alpha(t, \infty)\};$$

$$q_t = I - \sup\{\mathrm{rp}(y) : y \in A^\alpha(-\infty, -t)\};$$

$$g_t = \sup\{q_s : s < t\};$$

$$f_\infty = \sup\{f_t : t > 0\} \quad \text{and} \quad g_\infty = \sup\{g_t : t > 0\}.$$

Lemma 2.3. *For $t \geqq 0$ write*

$$r(t) = \sup\{\mathrm{rp}(y) : y \in A^\alpha(-\infty, -t]\};$$

and

$$l(t) = \sup\{\mathrm{rp}(y) : y \in A^\alpha[t, \infty)\}.$$

Then, for every $t \geqq 0$ and $s \geqq 0$,

$$r(s)M^\alpha([-t-s, \infty))l(t) \subseteq B.$$

Proof. Fix $z \in M^\alpha[-t-s, t+s]$ and $P \in \mathrm{lat}\, B \subseteq A'$. Then for every $x \in A^\alpha(-\infty, -s]$ and $y \in A^\alpha[t, \infty)$,

$$x^*yz \in M^\alpha[s, \infty), \qquad M^\alpha[-s-t, \infty)M^\alpha[t, \infty) \subseteq M^\alpha[0, \infty) \subseteq B.$$

Since x, y, $\mathrm{rp}(x)$ and $\mathrm{rp}(y)$ commute with P, we have

$$[x(I-P)(H)] = [(I-P)x(H)] = [(I-P)\mathrm{rp}(x)(H)] = [\mathrm{rp}(x)[(I-P)(H)]$$

and similarly $[yP(H)]=[\mathrm{rp}(y)P(H)]$. Note also that $[x(I-P)(H)]\subseteq(I-P)(H)$ and $[yP(H)]\subseteq P(H)$ since $x\in H^\infty(\alpha)^*\subseteq B^*$ and $y\in H(\alpha)\subseteq B$. Since $x^*zy\in B$ and $P\in\mathrm{lat}\,B$, the subspace $[x^*zyP(H)]$ is orthogonal to the subspace $(I-P)(H)$. Hence $[zyP(H)]$ is orthogonal to $[x(I-P)(H)]=[\mathrm{rp}(x)(I-P)(H)]$. But $[zyP(H)]=[z\,\mathrm{rp}(y)P(H)]$. Thus $[\mathrm{rp}(x)z\,\mathrm{rp}(y)P(H)]$ is orthogonal to $(I-P)(H)$. This shows that $\mathrm{rp}(x)z\,\mathrm{rp}(y)\in\mathrm{alg}\{P\}$ whenever $x\in A^\alpha(-\infty,-s]$ and $y\in A^\alpha[t,\infty)$. Hence $r(s)zl(t)\in\mathrm{alg}\{P\}$. Since this holds for every $P\in\mathrm{lat}\,B$ and $B=\mathrm{alg\,lat}\,B$, $r(s)zl(t)\in B$. ∎

Corollary 2.4. *For $t,s\geqq 0$ we have*

(1) $(I-g_s)M^\alpha(-t-s,\infty)(I-f_t)\subseteq B$;
(2) $(I-f_t)M^\alpha[0,t+s)(I-g_s)\subseteq A$;
(3) $(I-f_t)M^\alpha[0,t]\subseteq A$;
(4) $M^\alpha[0,t)(I-g_t)\subseteq A$;
(5) $M^\alpha[-t,\infty)(I-f_t)\subseteq B$;
(6) $(I-g_s)M^\alpha(-s,\infty)\subseteq B$.

Proof. It follows from Lemma 2.3 that $(I-q_r)M^\alpha[-t-r,\infty)(I-f_t)\subseteq B$ for every $r<s$. But then $(I-g_s)M^\alpha[-t-r,\infty)(I-f_t)\subseteq B$ for every $r<s$ (as $g_s\geqq q_r$). Hence (1) follows. Lemma 2.3 also implies (5) (set $s=0$ in Lemma 2.3) and (6) (set $t=0$). We then have

$$(I-f_t)M^\alpha[0,t+s)(I-g_s)\subseteq B^*\cap B=A.$$

This proves (2) and, similarly, (3) and (4) follow from (5) and (6). ∎

Lemma 2.5. (1) *For each $t\in\mathbf{R}$, f_t and g_t lie in $Z(M^\alpha)$ (the center of the fixed point algebra).*

(2) *For $t<0$, $f_t=0$ and, for $t\leqq 0$, $g_t=0$.*
(3) *If $t\leqq s$ then $f_t\leqq f_s$ and $g_t\leqq g_s$.*
(4) *$\wedge\{f_s : s>t\}=f_t$ and $\vee\{g_s : s<t\}=g_t$ for every $t\in\mathbf{R}$.*
(5) *$A^\alpha[s,\infty)(I-f_t)(H)\subseteq(I-f_{t+s})(H)$ and $A^\alpha(-\infty,-s](I-g_t)(H)\subseteq(I-g_{t+s})(H)$, $s,t\in\mathbf{R}$.*
(6) f_∞ and g_∞ lie in $Z(M^\alpha)\cap A'$.
(7) *For $s\geqq 0$ and $t\in\mathbf{R}$, $M^\alpha[-s,0](I-f_t)(H)\subseteq(I-f_{t-s})(H)$ and $M^\alpha[0,s](I-g_t)(H)\subseteq(I-g_{t-s})(H)$.*
(8) *$M^\alpha(-\infty,0](I-f_\infty)(H)\subseteq(I-f_\infty)(H)$ and $M^\alpha[0,\infty)(I-g_\infty)(H)\subseteq(I-g_\infty)(H)$.*
(9) *$(I-f_\infty)(I-g_\infty)$ lies in $Z(M)\cap M^\alpha$ and $(I-f_\infty)(I-g_\infty)M\subseteq A$.*

PROOF. (1) Clearly f_t and g_t lie in A. Since $A^\alpha(t,\infty)$ and $A^\alpha(-\infty,-t)$ are α-invariant, f_t and g_t lie in M^α. For every unitary operator $v \in M^\alpha$,

$$vA^\alpha(t,\infty)v^* = A^\alpha(t,\infty) \quad \text{and} \quad vA^\alpha(-\infty,-t)v^* = A^\alpha(-\infty,-t).$$

Hence f_t and g_t lie in $Z(M^\alpha)$. (2), (3) and (4) follow immediately from the definitions.

(5) Follows from the fact that

$$A^\alpha[s,\infty)A^\alpha(t,\infty) \subseteq A^\alpha(s+t,\infty)$$

and

$$A^\alpha(-\infty,-s]A^\alpha(-\infty,-t) \subseteq A^\alpha(-\infty,-s-t)$$

(the statement about g_t and g_{t+s} is first proved for q_t, q_{t+s}, and then $I-g_t = \wedge\{I-q_r : r<t\}$ is used).

(6) From (5) we have $A^\alpha[s,\infty)(\bigcap_t(I-f_t)(H)) \subseteq \bigcap_t(I-f_{t+s})(H) = \bigcap_t(I-f_t)(H)$ and $A^\alpha(-\infty,s](\bigcap_t(I-g_t)(H)) \subseteq \bigcap_t(I-g_t)(H)$ for every $s \in \mathbf{R}$. Hence $A^\alpha[s,\infty)(I-f_\infty)(H) \subseteq (I-f_\infty)(H)$ and $A^\alpha(-\infty,-s](I-g_\infty)(H) \subseteq (I-g_\infty)(H)$ for all $s \in \mathbf{R}$. Since both $\bigcup\{A^\alpha[s,\infty) : s \in \mathbf{R}\}$ and $\bigcup\{A^\alpha(-\infty,-s] : s \in \mathbf{R}\}$ are σ-weakly dense in A, f_∞ and g_∞ lie in A'.

(7) If $t<s$, $f_{t-s} = g_{t-s} = 0$ and statement (7) clearly holds. So assume $t \geqq s \geqq 0$. Then, for $x \in M^\alpha[-s,0]$ and $y \in A^\alpha(t,\infty)$, we have $xy \in B^*B^* \subseteq B^*$ and $xy \in M^\alpha[-s,0]M^\alpha(t,\infty) \subseteq M^\alpha(t-s,\infty) \subseteq B$; i.e. $xy \in A^\alpha(t-s,\infty)$. Hence $M^\alpha[-s,0](I-f_t)(H) \subseteq (I-f_{t-s})(H)$. The proof for g_t is similar.

(8) For $s \geqq 0$, it follows from part (7) that

$$M^\alpha[-s,0](I-f_\infty)(H) \subseteq (I-f_\infty)(H).$$

Since $\bigcup\{M^\alpha[-s,0] : s \geqq 0\}$ is σ-weakly dense in $M^\alpha(-\infty,0]$, $M^\alpha(-\infty,0](I-f_\infty)(H) \subseteq (I-f_\infty)(H)$. The proof for g_∞ is similar.

To prove (9) fix $t>0$ and let x be in $M^\alpha[0,t)$.

Using Corollary 2.4(3) we have $(I-f_t)x \in A$ and using Corollary 2.4(4), $(1-g_t)x \in A$. Write F for $(1-f_\infty)(1-g_\infty)$. Then

$$xF = x(1-g_t)F = Fx(1-g_t)F = FxF$$

and

$$Fx = F(1-f_t)x = F(1-f_t)xF = FxF..$$

Hence F commutes with $M^\alpha[0,t)$. Since this holds for every $t>0$, F commutes with $M^\alpha[0,\infty)$ and, as $M[0,\infty)+M[0,\infty)^*$ is σ-weakly dense in M, F commutes with M. Also, for $t>0$,

$$FM^\alpha[0,t] = F(1-f_t)M^\alpha[0,t] \subseteq A.$$

Hence $FM^\alpha[0,\infty) \subseteq A$ and by taking adjoints and using the fact that $F \in Z(M)$ we have

$$FM^\alpha(-\infty,0] = M^\alpha(-\infty,0]F = (FM[0,\infty))^* \subseteq A^* = A.$$

It follows that $MF \subseteq A$. ∎

We shall now assume that $(1-f_\infty)(1-g_\infty)=0$.

We define a projection valued measure $Q(\cdot)$ on $\mathbf{R}$, with values in $Z(M^\alpha)$, by

$$Q(t,\infty) = f_\infty(I-f_t) + (I-f_\infty)g_{1-t}, \qquad t \in \mathbf{R}.$$

Since $\vee\{Q(t,\infty): t \in \mathbf{R}\} = f_\infty + (I-f_\infty)g_\infty = I$, $\wedge\{Q(t,\infty): t \in \mathbf{R}\} = 0$ and

$$\begin{aligned}\vee\{Q(t,\infty): t > s\} &= f_\infty(I - \wedge\{f_t : t > s\}) + (I-f_\infty)\vee(g_{1-t} : t > s\}\\ &= f_\infty(I-f_s) + (I-f_\infty)g_{1-s} = Q(s,\infty),\end{aligned}$$

the measure $Q(\cdot)$ is well defined (see [15, 15.7] for a similar construction). We now define a one parameter group $U = \{U_t : t \in \mathbf{R}\}$ of unitary operators in $Z(M^\alpha)$ by

$$U_t = \int_{-\infty}^{\infty} e^{its} dQ(s), \qquad t \in \mathbf{R}.$$

For $t \in \mathbf{R}$ we let σ_t be the automorphism

$$\sigma_t(x) = U_t^* \alpha_t(x) U_t, \qquad x \in M.$$

This defines an action σ of $\mathbf{R}$ on M.

Proposition 2.6. $A \subseteq M^\sigma$.

Proof. For $X \in A$ let $\beta_t(x)$ be $U_t x U_t^*$. This defines an action of $\mathbf{R}$ on A. Using Lemma 2.5(5) we see that $A^\alpha[s,\infty)Q(t,\infty) \subseteq Q(t+s,\infty)$ for every s and t. It follows ([4, Theorem 2.9]) that, for $s \in \mathbf{R}$,

$$A^\alpha[s,\infty) \subseteq A^\beta[s,\infty).$$

Hence $\beta = \alpha$ on A and, consequently, $A \subseteq M^\sigma$. ∎

Lemma 2.7. *Let x be in M and I, J intervals of* $\mathbf{R}$ *(with closures $\bar{I}$, $\bar{J}$).*

(1) *If* $\mathrm{sp}_\alpha(x) \subseteq [a,b]$ *then* $\mathrm{sp}_\sigma(Q(I)xQ(J)) \subseteq -\bar{I} + [a,b] + \bar{J}$.

(2) *If* $\mathrm{sp}_\sigma(x) \subseteq [c,d]$ *then* $\mathrm{sp}_\alpha(Q(I)xQ(J)) \subseteq \bar{I} + [c,d] - \bar{J}$.

PROOF. The proof is a modification of [15, 15.12]. Write N for $M \otimes F_2$ where F_2 is the factor of type I_2 and identify it with the 2×2 matrices over M.

Let

$$\theta_t\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \alpha_t(x_{11}) & \alpha_t(x_{12})U_t \\ U_t^*\alpha_t(x_{21}) & \sigma_t(x_{22}) \end{pmatrix}$$

for $(x_{ij}) \in N$, $t \in \mathbf{R}$. Then θ defines an action of $\mathbf{R}$ on N. We have

$$\theta_t\begin{pmatrix} 0 & 0 \\ Q(I) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ U_t^*Q(I) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \exp(-itT)Q(I) & 0 \end{pmatrix}$$

where $T = \int_{-\infty}^{\infty} s\,dQ(s)$.

Hence, for every $h \in L^1(\mathbf{R})$,

$$\theta_h\begin{pmatrix} 0 & 0 \\ Q(I) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \hat{h}_t(-T)Q(I) & 0 \end{pmatrix}$$

where $\hat{h}$ is the Fourier transform of h. But $\hat{h}(-T)Q(I) = 0$ whenever $\operatorname{supp} \hat{h} \subseteq \mathbf{R} \setminus -\bar{I}$.

It follows that

$$\operatorname{sp}_\theta\begin{pmatrix} 0 & 0 \\ Q(I) & 0 \end{pmatrix} \subseteq \{t : \hat{h}(t) = 0 \text{ whenever } \operatorname{supp} \hat{h} \subseteq \mathbf{R} \setminus -I\} \subseteq -\bar{I}.$$

Similarly

$$\operatorname{sp}_\theta\begin{pmatrix} 0 & Q(J) \\ 0 & 0 \end{pmatrix} \subseteq \bar{J}.$$

Since

$$\begin{pmatrix} 0 & 0 \\ 0 & Q(I)xQ(J) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ Q(I) & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & Q(J) \\ 0 & 0 \end{pmatrix},$$

we have

$$\operatorname{sp}_\sigma(Q(I)xQ(J)) = \operatorname{sp}_\theta\begin{pmatrix} 0 & 0 \\ 0 & Q(I)xQ(J) \end{pmatrix} \subseteq -\bar{I} + [a, b] + \bar{J}.$$

This proves (1). The proof of (2) is similar. ∎

LEMMA 2.8. *Let C be an α-invariant σ-weakly closed subspace of M.*

(1) *C is generated by the elements of C with a compact Arveson's spectrum. In fact, there is a net $\{h_i\}$ of functions in $L^1(\mathbf{R})$ such that $\alpha_{h_i}(x) \to x$ σ-weakly for every $x \in M$ and* supp $\hat{h}_i$ *is compact for all i.*

(2) *Suppose* $x \in M$ *and* $h \in L^1(\mathbf{R})$ *such that* $\hat{h} = 1$ *on an open set* $W \subseteq \mathbf{R}$. *Then* $\mathrm{sp}_\alpha(x - \alpha_h(x)) \cap W = \varnothing$.

(3) *If* $x \in C$ *and* W *is an open subset of* $\mathbf{R}$ *which contains a compact set* F, *then* $x = x_1 + x_2$ *where* $x_i \in C$, $i = 1, 2$, $\mathrm{sp}_\alpha(x_1) \subseteq W \cap \mathrm{sp}_\alpha(x)$ *and* $\mathrm{sp}_\alpha(x_2) \subseteq (\mathbf{R} \setminus F) \cap \mathrm{sp}_\alpha(x)$.

(4) *Given* $x \in C$ *with* $\mathrm{sp}_\alpha(x)$ *compact and* $\varepsilon > 0$, *we can write* x *as a finite sum* $\Sigma_{i=1}^n x_i$ *where for every* i *there is some* $t_i \in \mathbf{R}$ *such that* $\mathrm{sp}_\alpha(x_i) \subseteq \mathrm{sp}_\alpha(x) \cap [t_i, t_i + \varepsilon]$.

(5) *Given* $x \in C$ *with* $\mathrm{sp}_\alpha(x)$ *compact*, $t \in \mathbf{R}$ *and* $\varepsilon > 0$ *we can write* $x = x_1 + x_2 + x_3$ *where* $x_i \in C$ *for* $i = 1, 2, 3$, $\| x_1 \| < 2 \| x \|$, $\mathrm{sp}_\alpha(x_1) \subseteq (t - \varepsilon, t + \varepsilon)$, $x_3 \in M^\alpha(-\infty, t)$, *and* $x_2 \in M^\alpha(t, \infty)$.

PROOF. The proof of (1) can be found in [15, Lemma 13.2].

(2) Suppose $\hat{h} = 1$ on W and $t \in \mathrm{sp}_\alpha(x - \alpha_h(x)) \cap W$. Then there is some $f \in L^1(\mathbf{R})$ with $\hat{f}(t) \neq 0$ and $\mathrm{supp}\, \hat{f} \subseteq W$. We have $\hat{h}\hat{f} = \hat{f}$ and thus $h * f = f$ and $\alpha_f(x - \alpha_h(x)) = 0$. Hence $t \in \{s : \hat{f}(s) = 0\}$ contradicting our choice of f.

(3) Using [12, Theorem 2.6.2] there is a function $h \in L^1(\mathbf{R})$ such that $\hat{h} = 1$ on F and $\mathrm{supp}\, \hat{h} \subseteq W$. Let x_1 be $\alpha_h(x)$ and x_2 be $x - x_1$. Then

$$\mathrm{sp}_\alpha(x_1) \subseteq \mathrm{supp}\, \hat{h} \cap \mathrm{sp}_\alpha(x) \subseteq W \quad \text{and} \quad \mathrm{sp}_\alpha(x_2) \subseteq \mathbf{R} \setminus F$$

by (2).

(4) Suppose $\mathrm{sp}_\alpha(x) \subseteq (a, b)$, $-\infty < a < b < \infty$. Let $W = (a - \varepsilon/2, a + \varepsilon/2)$ and $F = [a, a + \varepsilon/3]$ and apply (3) to get $x = x_1 + y$ where $\mathrm{sp}_\alpha(x_1) \subseteq (a, a + \varepsilon)$ and $\mathrm{sp}_\alpha(y) \subseteq (a + \varepsilon/3, b)$. Apply (3) again, to y, and continue by induction.

(5) Using [12, Theorem 2.6.3] there is a $h \in L^1(\mathbf{R})$ such that $\| h \| < 2$, $\mathrm{supp}\, \hat{h} \subseteq (t - \varepsilon, t + \varepsilon)$ and $\hat{h} = 1$ on some neighborhood of t, say $(t - \delta, t + \delta)$. Let $x_1 = \alpha_h(x)$ and $y = x - x_1$. Then $\mathrm{sp}_\alpha(y) \subseteq (-\infty, t - \delta] \cup [t + \delta, \infty)$. Apply (3) with $W = (0, \infty)$ and $F = [\delta, s]$ (where s is such that $\mathrm{sp}_\alpha(x) \subseteq (-\infty, s)$) to get $y = x_2 + x_3$ as desired. ■

LEMMA 2.9. $f_\infty M(1 - f_\infty) \subseteq M^\sigma(0, \infty)$.

PROOF. By Lemma 2.5(8), $f_\infty M^\alpha(-\infty, 0](1 - f_\infty) = \{0\}$. Since $M^\alpha[0, \infty) + M^\alpha(-\infty, 0]$ is σ-weakly dense in M [4, Theorem 3.15] it is left to show that $f_\infty M[0, \infty)(1 - f_\infty) \subseteq M^\sigma(0, \infty)$.

Fix $x \in f_\infty M^\alpha[0, \infty)(1 - f_\infty)$ and set $s \geqq 0$, $t \geqq 0$ and $0 < \varepsilon < \frac{1}{9}$. Using Lemma 2.8(3) we can write $x = x_1 + x_2$ where $x_1 \in M^\alpha[0, t + s - 3\varepsilon]$, $x_2 \in M^\alpha[t + s - \frac{1}{3}, \infty)$, $x_1 = f_\infty x_1(1 - f_\infty)$ and $x_2 = f_\infty x_2(1 - f_\infty)$. (If $t + s \leqq 3\varepsilon$, $x = x_2$.) Note that

$$f_t - f_{t-\varepsilon} = f_\infty Q(t-\varepsilon, t]$$

and

$$(1-f_\infty)(g_s - g_{s-\varepsilon}) = (1-f_\infty)Q(1-s, 1-s+\varepsilon].$$

Hence, using Lemma 2.7(1),

$$\begin{aligned} &\mathrm{sp}_\sigma((f_t - f_{t-\varepsilon})x_2(g_s - g_{s-\varepsilon})) \\ &\qquad \subseteq [-t, -t+\varepsilon] + [t+s-\tfrac{1}{3}, \infty) + [1-s, 1-s+\varepsilon] \subseteq [\tfrac{2}{3}, \infty). \end{aligned}$$

Hence $(f_t - f_{t-\varepsilon})x_2(g_s - g_{s-\varepsilon}) \in M^\sigma(0, \infty)$.

Also using Corollary 2.4(2),

$$(f_t - f_{t-\varepsilon})x_1(g_s - g_{s-\varepsilon}) \in (1 - f_{t-\varepsilon})M^\alpha[0, t+s-2\varepsilon)(1 - g_{s-\varepsilon}) \subseteq A.$$

But $f_\infty \in Z(A)$; hence $(f_t - f_{t-\varepsilon})x_1(g_s - g_{s-\varepsilon}) = 0$. Therefore

$$(f_t - f_{t-\varepsilon})x(g_s - g_{s-\varepsilon}) \subseteq M^\sigma(0, \infty).$$

Since $f_\infty = \Sigma_{k=0}^{\infty}(f_{k\varepsilon} - f_{(k-1)\varepsilon})$ and

$$1 - f_\infty = (1-f_\infty)g_\infty = (1-f_\infty)\,\Sigma_{m=0}^{\infty}(g_{m\varepsilon} - g_{(m-1)\varepsilon}),$$

$x \in M^\sigma(0, \infty)$. ∎

Proposition 2.10. $M^\alpha[0, \infty) \subseteq M^\sigma[0, \infty)$.

Proof. Let x be an element of $M^\alpha[0, \infty)$ for which $\mathrm{sp}_\alpha(x)$ is compact and write $x_1 = xf_\infty$, $x_2 = (1-f_\infty)x(1-f_\infty)$ and $x_3 = f_\infty x(1-f_\infty)$. We shall show that x_1, x_2 and x_3 lie in $M^\sigma[0, \infty)$; as $x = x_1 + x_2 + x_3$, this will complete the proof.

The fact that x_3 lies in $M^\sigma[0, \infty)$ follows from Lemma 2.9.

Now assume that y is an element of $M^\alpha[t, t+\varepsilon]$ for some $t \geqq 0$ and $\varepsilon > 0$ and $y = yf_\infty$. Then, by Corollary 2.4(3), $Q(t+\varepsilon, \infty)y = f_\infty[1 - f_{t+\varepsilon})y \in A$ and by Proposition 2.6, $Q(t+\varepsilon, \infty)y \in M^\sigma \subseteq M^\sigma[0, \infty)$. Using Lemma 2.7 and the fact that $y = yf_\infty = yQ[0, \infty)$, we have

$$\begin{aligned} \mathrm{sp}_\sigma(Q(-\infty, t+\varepsilon]y) &= \mathrm{sp}_\sigma(Q(-\infty, t+\varepsilon]yQ[0, \infty)) \\ &\subseteq [-t-\varepsilon, \infty) + [t, t+\varepsilon] + [0, \infty) \subseteq [-\varepsilon, \infty). \end{aligned}$$

Hence $y = Q(t+\varepsilon, \infty)y + Q(-\infty, t+\varepsilon]y \in M^\sigma[-\varepsilon, \infty)$.

Fix $\varepsilon > 0$. Since $\mathrm{sp}_\alpha(x_1)$ is compact we can write (Lemma 2.8(4)) $x_1 = \Sigma_{i=1}^{n} y_i$ where, for every i, $y_i = y_i f_\infty$ and $\mathrm{sp}_\alpha(y_i) \subseteq [t_i, t_i + \varepsilon]$ for some $t_i \geqq 0$. As we have just shown, $y_i \in M^\sigma[-\varepsilon, \infty)$. Hence $x_1 \in M^\sigma[-\varepsilon, \infty)$. But this holds for every $\varepsilon > 0$; hence $x_1 \in M^\sigma[0, \infty)$.

For x_2, consider first an element z of $M^\alpha[t, t+\varepsilon)$ (for some $t \geqq 0$, $\varepsilon > 0$) such that $z = (I - f_\infty)z(I - f_\infty)$. By Corollary 2.4(4), $z(1 - g_{t+\varepsilon}) \in A \subseteq M^\sigma \subseteq M^\sigma[0, \infty)$. We have

$$(I - f_\infty)Q(1 - t - \varepsilon, \infty) = (I - f_\infty)Q(1 - t - \varepsilon, \infty) = (I - f_\infty)g_{t+\varepsilon}$$

and

$$(I - f_\infty)Q(-\infty, 1] = (I - f_\infty)(I - Q(1, \infty)) = I - f_\infty.$$

Hence, using Lemma 2.7(1),

$$\begin{aligned} \mathrm{sp}_\sigma(zg_{t+\varepsilon}) &= \mathrm{sp}_\sigma(Q(-\infty, 1]zQ(1 - t - \varepsilon, \infty)) \\ &\subseteq [-1, \infty) + [t, t+\varepsilon] + [1 - t - \varepsilon, \infty] \subseteq [-\varepsilon, \infty). \end{aligned}$$

Therefore, for such z, $z \in M^\sigma[-\varepsilon, \infty)$. Now, fix $\varepsilon > 0$ and write $x_2 = \Sigma_{i=1}^m z_i$ where $z_i \in M^\alpha[t_i, t_i + \varepsilon)$ (for some $t_i \geqq 0$) and $z_i = (1 - f_\infty)z_i(1 - f_\infty)$. We have shown that each z_i lies in $M^\sigma[-\varepsilon, \infty)$ and, thus, x_2 lies in $M^\sigma[-\varepsilon, \infty)$. Since $\varepsilon > 0$ is arbitrary, $x_2 \in M^\sigma[0, \infty)$. ■

LEMMA 2.11. (1) $M^\sigma(0, \infty) \subseteq M^\alpha[0, \infty)$.
(2) $B \subseteq M^\sigma[0, \infty)$.

PROOF. (1) Let x be in $M^\sigma(0, \infty)$ and $h \in L^1(\mathbf{R})$ with supp $\hat{h} \subseteq (-\infty, 0]$. Then $\alpha_h(x) \in M^\sigma(0, \infty) \cap M^\alpha(-\infty, 0]$. (Note that, for a subset S of $\mathbf{R}$, $M^\sigma(S)$ is α-invariant since $\sigma_s\alpha_t = \alpha_t\sigma_s$ for all s, t.) Hence by Proposition 2.10,

$$\alpha_h(x)^* \in M^\sigma(-\infty, 0) \cap M^\alpha[0, \infty) \subseteq M^\sigma(-\infty, 0) \cap M^\sigma[0, \infty) = \{0\}.$$

Thus, $\alpha_h(x) = 0$ for every such h. It follows that $\mathrm{sp}_\alpha(x) \subseteq [0, \infty)$.

(2) Let x be in B and $h \in L^1(\mathbf{R})$ with supp $\hat{h} \subseteq (-\infty, 0)$. Note that, since B is α-invariant and $Z(M^\alpha) \subseteq B$, B is also σ-invariant. Hence $\sigma_h(x) \in B$. Also $\mathrm{sp}_\sigma(\sigma_h(x)) \subseteq \mathrm{supp}\, \hat{h} \subseteq (-\infty, 0)$. Hence $\sigma_h(x)^* \in M^\sigma(0, \infty)$ and, using part (1) and the fact that $M^\alpha[0, \infty) \subseteq B$, we have $\sigma_h(x)^* \in B$. Therefore $\sigma_h(x) \in B \cap B^* = A \subseteq M^\sigma$. But then $\mathrm{sp}_\sigma(\sigma_h(x)) \subseteq \{0\}$. Since $\mathrm{sp}_\sigma(\sigma_h(x)) \subseteq (-\infty, 0)$, $\mathrm{sp}_\sigma(\sigma_h(x)) = \varnothing$ and $\sigma_h(x) = 0$. Since this holds for every $h \in L^1(\mathbf{R})$ with supp $\hat{h} \subseteq (-\infty, 0)$, $x \in M^\sigma[0, \infty)$. ■

Now write R for M^σ and, for a subset $S \subseteq \mathbf{R}$, $R^\alpha(S)$ will denote $R \cap M^\alpha(S)$. Clearly R is α-invariant (and so are the spectral subspaces $R^\alpha(S)$). We also know that R contains A.

LEMMA 2.12. (1) $f_0 R^\alpha[0,\infty) \subseteq M^\alpha \subseteq A$ *and* $f_\infty \in Z(R)$.

(2) $(I - f_\infty)M^\alpha[0,\infty) \subseteq A$.

(3) $(I - f_\infty)R \subseteq A$.

(4) $(I - f_0)R^\alpha[0,\infty)(I - q_0) \subseteq A$.

PROOF. (1) Since $f_\infty M(I - f_\infty) \subseteq M^\alpha(0,\infty)$ (Lemma 2.9), we have $f_\infty R(1 - f_\infty) = \{0\}$. Hence $f_\infty \in Z(R)$ and, therefore, $f_0 R^\alpha[0,\infty) = f_0 R^\alpha[0,\infty) f_\infty$. For $x \in R^\alpha[0,\infty)$ we have (Lemma 2.7(2)),

$$\begin{aligned} \mathrm{sp}_\alpha(f_0 x) &= \mathrm{sp}_\alpha(f_0 x f_\infty) = \mathrm{sp}_\alpha(Q(0) f_\infty x f_\infty Q[0,\infty)) \\ &\subseteq \{0\} + \{0\} + (-\infty, 0] \subseteq (-\infty, 0]. \end{aligned}$$

Hence $f_0 x \in M^\alpha(-\infty, 0] \cap M^\alpha[0,\infty) = M^\alpha$.

(2) We have, for $t > 0$, $(I - f_\infty)M^\alpha[0,t] = (I - f_\infty)(I - f_t)M^\alpha[0,t] \subseteq A$ (Corollary 2.4(3)). Hence $(I - f_\infty)M^\alpha[0,\infty) \subseteq A$.

(3) From part (2) we have $(I - f_\infty)R^\alpha[0,\infty) \subseteq A$. Hence $R^\alpha(-\infty,0](I - f_\infty) \subseteq A$. But $f_\infty \in Z(R)$; hence $(I - f_\infty)R^\alpha(-\infty,0] \subseteq A$. Now, R is a von Neumann algebra and it is the σ-weak closure of $R^\alpha[0,\infty) + R^\alpha(-\infty,0]$ ([4, Theorem 3.15]). Thus $(I - f_\infty)R \subseteq A$.

(4) Fix $x \in R^\alpha[0,\infty)$. For $t \geqq 0$ and $s > 2\varepsilon > 0$ write $x = x_1 + x_2$ where $x_1 \in R^\alpha[0, s+t)$ and $x_2 \in R^\alpha[s+t-\varepsilon, \infty)$ (Lemma 2.8(3)). Then $(f_{t+\varepsilon} - f_t)x_1(I - g_s) \in A$ (Corollary 2.4(2)) and

$$\begin{aligned} \mathrm{sp}_\sigma((f_{t+\varepsilon} - f_t)x_2(I - g_s)) &= \mathrm{sp}_\sigma(Q(t, t+\varepsilon] f_\infty x_2 f_\infty Q[0,\infty)(1 - g_s)) \\ &\subseteq [-t-\varepsilon, -t] + [s+t-\varepsilon, \infty) + [0,\infty) \subseteq [s - 2\varepsilon, \infty) \subseteq (0,\infty). \end{aligned}$$

Since $\mathrm{sp}_\sigma((f_{t+\varepsilon} - f_t)x_2(I - g_s)) \subseteq \{0\}$, $(f_{t+\varepsilon} - f_t)x_2(I - g_s) = 0$. Hence, whenever $t \geqq 0$ and $s > 2\varepsilon > 0$,

$$(f_{t+\varepsilon} - f_t)x(I - g_s) \in A.$$

As $f_\infty - f_0 = \Sigma_{k=0}^\infty f_{(k+1)\varepsilon} - f_{k\varepsilon}$,

$$(f_\infty - f_0)x(I - g_s) \in A \qquad \text{for every } s > 0.$$

As $\inf\{g_s : s > 0\} = q_0$, $(f_\infty - f_0)x(I - q_0) \in A$.

From part (3), $(I - f_\infty)x \in A$. Therefore $(I - f_0)x(I - g_0) \in A$. ∎

PROPOSITION 2.13. (1) $(I - q_0)M^\sigma[0,\infty)(I - f_0) \subseteq B$.

(2) $M^\sigma[0,\infty) f_0 \subseteq M^\alpha[0,\infty) \subseteq B$.

PROOF. (1) Let $x = (I - q_0)x(I - f_0) \in M^\sigma[0,\infty) \cap M^\alpha[-t,t]$ for some

$t > 0$. By Lemma 2.8(5) we can, for every $\varepsilon > 0$, write $x = x_1(\varepsilon) + x_2(\varepsilon) + x_3(\varepsilon)$ where

(i) $x_i(\varepsilon) = (I - q_0)x_i(\varepsilon)(I - f_0) \in M^\sigma[0, \infty)$ for $i = 1, 2, 3$,

(ii) $\| x_1(\varepsilon) \| < 2 \| x \|$ and $x_1(\varepsilon) \in M^\alpha(-\varepsilon, \varepsilon)$,

(iii) $x_2(\varepsilon) \in M^\alpha(0, \infty)$ and $x_3(\varepsilon) \in M^\alpha(-\infty, 0)$.

Then $x_2(\varepsilon) \in B$. Also

$$x_3(\varepsilon) \in M^\alpha(-\infty, 0) \cap M^\sigma[0, \infty) \subseteq M^\sigma(-\infty, 0] \cap M^\sigma[0, \infty) = R.$$

Hence $x_3(\varepsilon) \in (I - q_0)R^\alpha(-\infty, 0)(I - f_0) \subseteq A$ (Lemma 2.12(4)).

By Corollary 2.4(5), we have $x_1(\varepsilon)(1 - f_\varepsilon) \in B$. Hence

$$x - x_1(\varepsilon)(f_\varepsilon - f_0) = x_2(\varepsilon) + x_3(\varepsilon) + x_1(\varepsilon)(1 - f_\varepsilon) \in B.$$

Since $\| x_1(\varepsilon) \| < 2 \| x \|$ for every $\varepsilon > 0$ and $f_\varepsilon - f_0 \to 0$ σ-weakly as $\varepsilon \to 0$, $x_1(\varepsilon)(f_\varepsilon - f_0) \to 0$ σ-weakly and, therefore, $x \in B$.

(2) Let x be in $M^\sigma[0, \infty)f_0$. Then $(I - f_\infty)x \in (I - f_\infty)M^\sigma[0, \infty)f_\infty = 0$ (Lemma 2.9). Hence $x = f_\infty x f_0$ and

$$\begin{aligned} \mathrm{sp}_\alpha(x) &= \mathrm{sp}_\alpha(f_\infty x f_0) = \mathrm{sp}_\alpha(Q[0, \infty)xQ(0)) \\ &\subseteq [0, \infty) + [0, \infty) - \{0\} \subseteq [0, \infty). \end{aligned}$$

Thus $x \in M^\alpha[0, \infty)$. ∎

Now write $\tilde{\alpha}$ for the action of $\mathbf{R}$ on M defined by

$$\tilde{\alpha}_t = \alpha_{-t}, \qquad t \in \mathbf{R}.$$

Since $B^* \supseteq M^\alpha(-\infty, 0] = M^{\tilde{\alpha}}[0, \infty)$, everything that was done in this section for B and α can be applied to B^* and $\tilde{\alpha}$. To do so note that, for $t \in \mathbf{R}$,

$$A^\alpha(t, \infty) = A^{\tilde{\alpha}}(-\infty, -t).$$

Instead of f_t and g_t we shall now have

$$\tilde{f}_t = I - \sup\{\mathrm{rp}(y) : y \in A^{\tilde{\alpha}}(t, \infty)\} \quad (= q_t),$$

$$\tilde{q}_t = I - \sup\{\mathrm{rp}(y) : y \in A^{\tilde{\alpha}}(-\infty, -t)\} \quad (= f_t),$$

$$\tilde{g}_t = \sup\{\tilde{q}_s : s < t\} \quad (= \sup\{f_s : s < t\}).$$

As in the discussion preceding Proposition 2.6, we let

$$\tilde{Q}(t, \infty) = \tilde{f}_\infty(I - \tilde{f}_t) + (I - \tilde{f}_\infty)\tilde{g}_{1-t} \quad (= g_\infty(I - q_t) + (I - g_\infty)\tilde{g}_{1-t})$$

and

$$\tilde{U}_t = \int_{-\infty}^{\infty} e^{its} d\tilde{Q}(s).$$

Then action $\tilde{\sigma}$ will now be defined by

$$\tilde{\sigma}_t(x) = \tilde{U}_t^* \alpha_{-t}(x) \tilde{U}_t.$$

Finally define the action θ of $\mathbf{R}$ on M by

$$\theta_t = \tilde{\sigma}_{-t}.$$

COROLLARY 2.14. (1) $M^\alpha[0, \infty) \subseteq M^\theta[0, \infty)$.

(2) $M^\theta(0, \infty) \subseteq M^\alpha[0, \infty)$.

(3) $B \subseteq M^\theta[0, \infty)$.

(4) $q_0 M^\theta[0, \infty) \subseteq B$.

PROOF. (1) Proposition 2.10, when applied to $\tilde{\alpha}$ and B^* (in place of α and B), implies $M^{\tilde{\alpha}}[0, \infty) \subseteq M^{\tilde{\sigma}}[0, \infty)$. Hence

$$\begin{aligned} M^\alpha[0, \infty) &= (M^\alpha(-\infty, 0]]^* = (M^{\tilde{\alpha}}[0, \infty))^* \\ &\subseteq (M^{\tilde{\sigma}}[0, \infty))^* = (M^\theta(-\infty, 0])^* = M^\theta[0, \infty). \end{aligned}$$

(2) Follows similarly from Lemma 2.11(1). For (3) note that Lemma 2.11(2), applied to $\tilde{\alpha}$ and B^*, implies that $B^* \subseteq M^{\tilde{\sigma}}[0, \infty) = M^\theta(-\infty, 0]$. Hence $B \subseteq M^\theta[0, \infty)$.

Part (4) follows similarly from Proposition 2.13(2) (noting that $\tilde{f}_0 = q_0$). ∎

LEMMA 2.15. *For all $s, t \in \mathbf{R}$,*

$$\sigma_t \circ \theta_s = \theta_s \circ \sigma_t.$$

PROOF. For $x \in M$ and t, s in $\mathbf{R}$,

$$\begin{aligned} \sigma_t(\theta_s(x)) &= \sigma_t(\tilde{U}_{-s}^* \alpha_s(x) \tilde{U}_{-s}) = U_t^* \alpha_t(\tilde{U}_{-s}^* \alpha_s(x) \tilde{U}_{-s}) U_t \\ &= U_t^* \tilde{U}_{-s}^* \alpha_{t+s}(x) \tilde{U}_{-s} U_t = \tilde{U}_{-s}^* U_t^* \alpha_{t+s}(x) U_t \tilde{U}_{-s} = \theta_s(\sigma_t(\sigma_t(x)). \end{aligned}$$ ∎

The next result (Proposition 2.16) might be known but I was unable to find a reference for it.

PROPOSITION 2.16. *Let $\theta = \{\theta_t : t \in \mathbf{R}\}$ and $\sigma = \{\sigma_t : t \in \mathbf{R}\}$ be two continuous actions of $\mathbf{R}$ on M that commute; i.e. $\theta_t \sigma_s = \sigma_s \theta_t$ for all t, s in $\mathbf{R}$. Define $\beta_t = \theta_t \circ \sigma_t$. Then*

(1) $\beta = \{\beta_t : t \in \mathbf{R}\}$ is a continuous action of $\mathbf{R}$ on M.

(2) For every a, b in $\mathbf{R}$,

$$M^{\sigma}[a,\infty)\cap M^{\theta}[b,\infty)\subseteq M^{\beta}[a+b,\infty).$$

PROOF. Since θ and σ commute (1) is obvious. (2) Define an action $\rho=\{\rho_{(t,s)}:(t,s)\in\mathbf{R}^2\}$ of $\mathbf{R}^2$ on M by $\rho_{(t,s)}=\theta_t\sigma_s$. Take $x\in M^{\sigma}[a,\infty)\cap M^{\theta}[b,\infty)$. Then there is a net $\{h_i\}$ of functions in $L^1(\mathbf{R})$ with supp $\hat{h}_i\subseteq[a,\infty)$ such that $\sigma_{h_i}(x)\rightarrow x$ σ-weakly. Since σ and θ commute, $M^{\theta}[b,\infty)$ is σ-invariant. Thus $\sigma_{h_i}\in M^{\theta}[b,\infty)$ for every i. Hence for every i there is a net $\{k_{ij}\}$ in $L_1(\mathbf{R})$ with supp $\hat{k}_{ij}\subseteq[b,\infty)$ such that

$$\theta_{k_{ij}}(\alpha_{h_i}(x))\underset{j}{\rightarrow}\sigma_{h_i}(x)$$

σ-weakly for for every i. It is, therefore, enough to assume that $x=\theta_k\sigma_h(y)$ for some $y\in M$, $k,h\in L^1(\mathbf{R})$ with supp $\hat{h}\subseteq[a,\infty)$ and supp $\hat{h}\subseteq[b,\infty)$ and prove that $x\in M^{\beta}[a+b,\infty)$.

For such x,

$$x=\int\int k(t)h(s)\theta_t\sigma_s(y)dtds=\int\int k(t)h(s)\rho_{(t,s)}(y)dtds=\rho_g(y)$$

where $g(t,s)=k(t)h(t)$. (Clearly $g\in L^1(\mathbf{R}^2)$.) For $(p,q)\in\mathbf{R}^2$ we have

$$\hat{g}(p,q)=\int\int g(t,s)e^{isq}e^{itp}dsdt=\int\int h(s)k(t)e^{itp}e^{isq}dsdt=\hat{h}(q)\hat{k}(p).$$

Hence supp $\hat{g}\subseteq[a,\infty)\times[b,\infty)$ and thus $x=\rho_g(y)\in M^{\rho}([a,\infty)\times[b,\infty))$.

Now let f be in $L^1(\mathbf{R})$ with supp $\hat{f}\subseteq(-\infty,a+b)$. For every $L>0$ define

$$f_L(t,s)=\begin{cases}f((t+s)/2), & |t-s|\leqq 2L,\\ 0, & \text{otherwise.}\end{cases}$$

Then $f_L\in L^1(\mathbf{R}^2)$. For $(p,q)\in\mathbf{R}^2$,

$$\hat{f}_L(p,q)=\int\int e^{ipt}e^{iqs}f_L(t-s)dtds.$$

Write $w=\frac{1}{2}(t+s)$ and $v=\frac{1}{2}(t+s)$ and get

$$\hat{f}_L(p,q)=\frac{1}{2}\int_{-L}^{L}\int_{-\infty}^{\infty}e^{ip(w+v)}e^{iq(w-v)}f(w)dwdv$$

$$=\frac{1}{2}\int_{-L}^{L}e^{-iv(q-p)}\left[\int_{-\infty}^{\infty}e^{iw(p+q)}f(w)dw\right]dv$$

$$=\frac{1}{2}\hat{f}(p+q)\int_{-L}^{L}e^{-iv(q-p)}dv.$$

Hence $\operatorname{supp}\hat{f}_L \subseteq \{(p,q): p+q<a+b\}$. In particular $\operatorname{sp}_\rho(x)\cap\operatorname{supp}\hat{f}_L = \varnothing$. Hence $\rho_{f_L}(x)=0$. But

$$0=\rho_{f_L}(x)=\int\int f_L(t,s)\theta_t\sigma_s(x)dtds$$

$$=\frac{1}{2}\int_{-L}^{L}\int_{-\infty}^{\infty} f(w)\theta_{w+v}\sigma_{w-v}(x)dwdv$$

$$=\frac{1}{2}\int_{-L}^{L}\int_{-\infty}^{\infty}\theta_v\sigma_{-v}\left(\int_{-\infty}^{\infty} f(w)\beta_w(x)dw\right)dv$$

$$=\frac{1}{2}\int_{-L}^{L}\theta_v\sigma_{-v}\beta_f(x)dv.$$

Hence, for every $L>0$

$$\frac{1}{L}\int_{-L}^{L}\theta_v\sigma_{-v}(\beta_f(x))dv=0.$$

Taking the limit as $L\to 0$ we have $\beta_f(x)=0$. Since f was arbitrary in $L^1(\mathbf{R})$ with $\operatorname{supp}\hat{f}\subseteq(-\infty,a+b)$, $\operatorname{sp}_\beta(x)\subseteq[a+b,\infty)$. ∎

Let σ be the action defined preceding Proposition 2.6 and θ be the action defined preceding Corollary 2.14. Let β be defined as in Proposition 2.16; i.e. $\beta_t=\sigma_t\theta_t$. Then, by Proposition 2.16,

$$M^\theta[b,\infty)\cap M^\sigma[a,\infty)\subseteq M^\beta[a+b,\infty).$$

But we also have $\theta_t=\beta_t\sigma_{-t}$ and $\sigma_t=\beta_t\theta_{-t}$ and β commutes with both $t\mapsto\sigma_{-t}$ and $t\mapsto\theta_{-t}$. Hence we can apply Proposition 2.16 to get the following.

COROLLARY 2.17. (1) $M^\beta[a,\infty)\cap M^\sigma(-\infty,-b]\subseteq M^\theta[a+b,\infty)$.
(2) $M^\beta[a,\infty)\cap M^\theta(-\infty,-b]\subseteq M^\sigma[a+b,\infty)$.

PROPOSITION 2.18. $M^\beta[0,\infty)=M^\sigma[0,\infty)\cap M^\theta[0,\infty)$.

PROOF. We know that $M^\sigma[0,\infty)\cap M^\theta[0,\infty)\subseteq M^\beta[0,\infty)$ (Proposition 2.16). Let x be in $M^\beta[0,\infty)$ and let h be in $L^1(\mathbf{R})$ with $\operatorname{supp}\hat{h}\subseteq(-\infty,0)$. Then

$$\sigma_h(x)\in M^\beta[0,\infty)\cap M^\sigma(-\infty,-b]$$

for some $b>0$. But, using Corollary 2.17(1), $\sigma_h(x)$ lies in $M^\theta[b,\infty)\subseteq M^\theta(0,\infty)$. Using Corollary 2.14(2), $\sigma_h(x)\in M^\alpha[0,\infty)$. But we also have $\sigma_h(x)\in M^\sigma(-\infty,0)\subseteq M^\alpha(-\infty,0]$. Hence $\sigma_h(x)\in M^\alpha\cap M^\sigma(-\infty,0)=\{0\}$. Since this holds for every $h\in L^1(\mathbf{R})$ with supp $\hat{h}\subseteq(-\infty,0)$, $x\in M^\sigma[0,\infty)$. Similarly we can prove that $M^\beta[0,\infty)$ is contained in $M^\theta[0,\infty)$. ■

We are now ready to prove the main result of this section.

THEOREM 2.19. *Let α be a continuous action of $\mathbf{R}$ on M and let B be a σ-weakly closed subalgebra of M that contains $M^\alpha[0,\infty)$ $(=H^\infty(\alpha))$. Then there is a projection $F\in M^\alpha\cap Z(M)$ and a strongly continuous one parameter unitary group $\{v_t: t\in\mathbf{R}\}$ in $Z(M^\alpha)$ such that*

(i) *$BF=MF$; and*

(ii) *$B(I-F)=M^\gamma[0,\infty)(I-F)$ where*

$$\gamma_t(x)=v_t^*\alpha_t(x)v_t,\qquad t\in\mathbf{R},\quad x\in M.$$

PROOF. Let F be $(I-f_\infty)(I-g_\infty)$. Then F lies in $M^\alpha\cap Z(M)$ and $BF\subseteq MF\subseteq AF\subseteq BF$ (Lemma 2.5(9)). This proves (i). Now we can assume $F=0$.

Let σ, θ and β be as defined above. Then

(1) $(I-q_0)M^\beta[0,\infty)(I-f_0)\subseteq(I-q_0)M^\sigma[0,\infty)(I-f_0)\subseteq B$ (Proposition 2.18 and Proposition 2.13(1));

(2) $M^\beta[0,\infty)f_0\subseteq M^\sigma[0,\infty)f_0\subseteq B$ (Proposition 2.18 and Proposition 2.13(2));

(3) $q_0M^\beta[0,\infty)\subseteq q_0M^\theta[0,\infty)\subseteq B$ (Proposition 2.18 and Corollary 2.14(4)).

Hence $M^\beta[0,\infty)\subseteq B$. On the other hand, $B\subseteq M^\sigma[0,\infty)\cap M^\theta[0,\infty)=M^\beta[0,\infty)$ (Lemma 2.11(2), Corollary 2.14(3) and Proposition 2.18). Therefore

$$M^\beta[0,\infty)=B.$$

We now write $\gamma_t=\beta_{t/2}$. This clearly defines a continuous action of $\mathbf{R}$ on M and $M^\gamma[0,\infty)=M^\beta[0,\infty)=B$.

We also have, for $t\in\mathbf{R}$ and $x\in M$,

$$\gamma_t(x)=\beta_{t/2}(x)=\sigma_{t/2}(\theta_{t/2}(x))=U_{t/2}^*\tilde{U}_{t/2}^*\alpha_t(x)\tilde{U}_{t/2}U_{t/2}.$$

Write $v_t=\tilde{U}_{t/2}U_{t/2}$ to complete the proof. ■

As a corollary we can derive the following result which was proved in [3] using different techniques.

Recall first that a subalgebra C of a von Neumann algebra M is called a *nest subalgebra* of M if there is a nest $\mathfrak{N}$ of projections of M such that

$$C = \{x \in M : (I - P)xP = 0 \text{ for all } P \in \mathfrak{N}\}.$$

In [4] the nest subalgebras of M were characterized as the analytic subalgebras $H^\infty(\alpha)$ of M associated with an *inner* action α of $\mathbf{R}$ on M (i.e. for every $t \in \mathbf{R}$, α_t is an inner automorphism). The following corollary now follows immediately from Theorem 2.19.

COROLLARY 2.20 ([3]). *If B is a σ-weakly closed subalgebra of M that contains a nest subalgebra C of M then B is a nest subalgebra of M.*

3. The maximality of $H^\infty(\alpha)$

The main result of this section (Theorem 3.7) proves that (under the assumption that $Z(M) \cap M^\alpha = \mathbf{C}I$) $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M if and only if either $\mathrm{sp}(\alpha) = \Gamma(\alpha)$ (i.e. Arveson's spectrum of α equals Connes spectrum) or there is a projection $P \in M$ such that $H^\infty(\alpha) = \{x \in M : (1 - P)xP = 0\}$.

As in Section 2, α is assumed to be a continuous action of $\mathbf{R}$ on a σ-finite von Neumann algebra M. If $0 \neq e \in M^\alpha$ is a projection then α defines a continuous action α^e of $\mathbf{R}$ on eMe by

$$\alpha_t^e = \alpha_t \,|\, eMe, \qquad t \in \mathbf{R}.$$

Connes' spectrum of α is defined to be

$$\Gamma(\alpha) = \bigcap \{\mathrm{sp}(\alpha^e) : e \text{ is a non-zero projection in } Z(M^\alpha)\}.$$

It is known that $\Gamma(\alpha)$ is a closed subgroup of $\mathbf{R}$ ([15, Proposition 16.1]). Thus either $\Gamma(\alpha) = \{0\}$ or $\Gamma(\alpha) = \mathbf{R}$ or $\Gamma(\alpha) = \{n\lambda : n \in \mathbf{Z}\}$ for some $\lambda \in \mathbf{R}$.

PROPOSITION 3.1. *Assume $Z(M) \cap M^\alpha = \mathbf{C}I$. If $\mathrm{sp}(\alpha) = \Gamma(\alpha)$ then $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M.*

PROOF. If $\mathrm{sp}(\alpha) = \Gamma(\alpha) = \{0\}$ then $H^\infty(\alpha) = M^\alpha = M$ and clearly $H^\infty(\alpha)$ is maximal. Suppose that there is some $\lambda > 0$ such that $\mathrm{sp}(\alpha) = \Gamma(\alpha) = \lambda\mathbf{Z}$. Then Proposition 16.4 of [15] implies that $Z(M^\alpha) = Z(M) \cap M^\alpha$. Hence M^α is a factor (since we assume $Z(M) \cap M^\alpha = \mathbf{C}I$). If B is a σ-weakly closed subalgebra of M containing $H^\infty(\alpha)$ and $\{f_t, g_s\}$ are the projection (in $Z(M^\alpha)$) associated with B as in Definition 2.2 then $\{f_t, g_s\} \subseteq \{0, I\}$. Hence $\{U_t : t \in \mathbf{R}\} \subseteq \mathbf{C}I$.

Therefore, if $(I - f_\infty)(I - g_\infty) = 0$, $\sigma = \alpha$ and $B = M^\sigma[0, \infty) = H^\infty(\alpha)$. If

$(I - f_\infty)(I - g_\infty) \neq 0$ then $(I - f_\infty)(I - g_\infty) = I$ and $M = B \cap B^*$; i.e. $B = M$ (Lemma 2.5(9)).

We now assume that $\mathrm{sp}(\alpha) = \Gamma(\alpha) = \mathbf{R}$. Suppose $t \geqq 0$ is such that $f_t \neq 0$ and $s > 0$ is arbitrary. From Lemma 2.5(7) we have

$$M^\alpha[-s, 0](I - f_{t+s})(H) \subseteq (I - f_t)(H).$$

Hence,

$$(*) \qquad (I - f_{t+s})M^\alpha[0, s]f_t = \{0\}.$$

Since $Z(M) \cap M^\alpha = \mathbf{C}I$, $[Mf_t(H)] = H$. Assume that $f_{t+s} \neq I$. Then $(I - f_{t+s})Mf_t \neq \{0\}$. Fix $\varepsilon > 0$. Then, there is some $r \in \mathbf{R}$ and $y \in M^\alpha(r - \varepsilon, r)$ such that $(1 - f_{t+s})yf_t \neq \{0\}$. Write e for the projection onto $[(I - f_{t+s})M^\alpha(r - \varepsilon, r)f_t(H)]$.

Since $\frac{1}{2}s - r \in \Gamma(\alpha) \subseteq \mathrm{sp}(\alpha^e)$, we have, for every $\delta > 0$,

$$eM^\alpha(\tfrac{1}{2}s - r - \delta, \tfrac{1}{2}s - r + \delta)e \neq \{0\}.$$

Hence

$$\begin{aligned} 0 &\neq [eM^\alpha(\tfrac{1}{2}s - r - \delta, \tfrac{1}{2}s - r + \delta)(I - f_{t+s})M^\alpha(r - \varepsilon, r)f_t(H)] \\ &\subseteq [eM^\alpha(\tfrac{1}{2}s - \delta - \varepsilon, \tfrac{1}{2}s + \delta)f_t(H)] \\ &\subseteq [(I - f_{t+s})M^\alpha(\tfrac{1}{2}s - \delta - \varepsilon, \tfrac{1}{2}s + \delta)f_t(H)]. \end{aligned}$$

Hence, for every $\varepsilon > 0$ and $\delta > 0$,

$$(I - f_{t+s})M^\alpha(\tfrac{1}{2}s - \delta - \varepsilon, \tfrac{1}{2}s + \delta)f_t \neq \{0\}.$$

By choosing ε and δ small enough we get a contradiction to (*). Therefore, if $f_t \neq 0$, then $f_{t+s} = I$ for every $s > 0$. But $f_t = \wedge\{f_{t+s} : s > 0\}$. Hence $\{f_t\} \subseteq \mathbf{C}I$. Similarly, $\{q_s\} \subseteq \mathbf{C}I$ and, therefore, $\{g_s\} \subseteq \mathbf{C}I$. Hence $\{U_t : t \in \mathbf{R}\} \subseteq \mathbf{C}I$. This shows that $B = M$ (if $(I - f_\infty)(I - g_\infty) = I$) or $B = H^\infty(\alpha)$ (if $(I - f_\infty)(I - g_\infty) = 0$). ∎

In order to prove the converse of this proposition we shall have to construct σ-weakly closed subalgebras of M that contain $H^\infty(\alpha)$. We shall construct such an algebra for every non-zero projection $e \in Z(M^\alpha)$.

Let e be a non-zero projection in $Z(M^\alpha)$. Write e_t for the projection onto $[M^\alpha[0, t]e(H)]$ ($e_t = 0$ if $t < 0$) and define f_t to be $\wedge\{e_s : s > t\}$. Also define g_t to be $I - \vee\{f_s : s \geqq 0\}$ if $t > 0$ and $g_t = 0$ if $t \leqq 0$. Write f_∞ for $\vee\{f_s : s \geqq 0\}$ and note that this is the projection onto $[M^\alpha[0, \infty)e(H)]$.

LEMMA 3.2. *For $\{f_t, g_t : t \in \mathbf{R}\}$ as defined above, we have the following*:

(1) *For each* $t \in \mathbf{R}$, f_t *and* g_t *lie in* $Z(M^\alpha)$.
(2) *If* $t \leqq s$ *then* $f_t \leqq f_s$ *and* $g_t \leqq g_s$.
(3) $\wedge\{f_t : t > s\} = f_s$ *and* $\vee\{g_t : t < s\} = g_s$ *for every* $s \in \mathbf{R}$.
(4) *For* $s \geqq 0$ *and* $t \in \mathbf{R}$, $M^\alpha[-s, 0](I - f_t)(H) \subseteq (I - f_{t-s})(H)$ *and* $M^\alpha[0, s](I - g_t)(H) \subseteq (I - g_{t-s})(H)$.
(5) $M^\alpha[0, \infty) f_\infty(H) \subseteq f_\infty(H)$.

PROOF. (1), (2), (3) and (5) are immediate. For (4) note that

$$M^\alpha[0, s][M^\alpha[0, w]e(H)] \subseteq [M^\alpha[0, s + w]e(H)].$$

Hence $M^\alpha[0, s]e_w(H) \subseteq e_{s+w}(H)$ for every $w > t - s$. Hence $M^\alpha[0, s] f_{t-s}(H) \subseteq f_t(H)$. It follows that

$$M^\alpha[0, s](I - f_t)(H) \subseteq (I - f_{t-s})(H).$$

The statement about (g_t) is immediate. ■

Let $\{f_t, g_t : t \in \mathbf{R}\}$ as above and define

$$Q(t, \infty) = f_\infty(I - f_t) + (I - f_\infty)g_{1-t}, \qquad t \in \mathbf{R},$$

and

$$U_t = \int_{-\infty}^{\infty} e^{its} dQ(s), \qquad t \in \mathbf{R}.$$

This defines an action σ of $\mathbf{R}$ on M by

$$\sigma_t(x) = U_t^* \alpha_t(x) U_t, \qquad t \in \mathbf{R}, \quad x \in M.$$

Note that for these action σ and measure $Q(\cdot)$, Lemma 2.7 is still valid.

LEMMA 3.3. *Let* e *and* $\{f_t : t \in \mathbf{R}\}$ *be as above.*
(1) *For every* $s \geqq t$ *and* $\delta > 0$

$$(f_s - f_t)(H) \subseteq [M^\alpha[t - 2\delta, s + \delta]e(H)].$$

(2) *For every* $a \leqq b < t \leqq s$,

$$(f_s - f_t)M^\alpha[a, b](I - f_{s-a}) = \{0\}.$$

(3) *For every* $b \geqq 0$

$$(f_\infty - f_b)M^\alpha[0, b](I - f_\infty) = \{0\}.$$

PROOF. (1) Fix $s \geqq t$ and $\delta > 0$. If $s < 0$ there is nothing to prove (as $f_s = f_t = 0$). If $s \geqq 0 > t$ then

$$\begin{aligned}(f_s - f_t)(H) = f_s(H) &\subseteq e_{s+\delta}(H)\\ &= [M^\alpha[0, s+\delta]e(H)]\\ &\subseteq [M^\alpha[t-2\delta, s+\delta]e(H)].\end{aligned}$$

Hence we now assume that $t \geqq 0$. For every integer $n \geqq 0$ we write $F(n, \delta)$ for the projection onto $[M^\alpha[n\delta, (n+2)\delta]e(H)]$. Then $F(n, \delta)$ lies in $Z(M^\alpha)$. If x lies in $M^\alpha[0, k\delta]$ for some integer $k \geqq 0$ then we can write

$$x = \sum_{i=0}^{k-2} x_i \qquad \text{where } \mathrm{sp}_\alpha(x_i) \subseteq [i\delta, (i+2)\delta].$$

Hence $[M^\alpha[0, k\delta]e(H)] = \vee\{F(n, \delta)(H) : 0 \leqq n \leqq k-2\}$. We now have, for every $m \geqq k \geqq 0$,

$$\begin{aligned}e_{(m+1)\delta} - e_{k\delta} &= \vee\{F(n, \delta) : 0 \leqq n \leqq m-1\} - \vee\{F(n, \delta) : 0 \leqq n \leqq k-2\}\\ &\leqq \vee\{F(n, \delta) : k-1 \leqq n \leqq m-1\}.\end{aligned}$$

Therefore $(e_{(m+1)\delta} - e_{k\delta})(H) \subseteq [M^\alpha[(k-1)\delta, (m+1)\delta]e(H)]$.

Since $s \geqq t \geqq 0$ there are non-negative integers k and m such that $m \geqq k$, $k\delta \leqq t < (k+1)\delta$ and $m\delta \leqq s < (m+1)\delta$. Then, $f_s \leqq e_{(m+1)\delta}$ and $f_t \geqq e_{k\delta}$ and, thus,

$$(f_s - f_t)(H) \subseteq [M^\alpha[(k-1)\delta, (m+1)\delta]e(H)] \subseteq [M^\alpha[t-2\delta, s+\delta]e(H)].$$

(2) Fix $a \leqq b < t \leqq s$. For every $0 < \delta < \frac{1}{2}(t-b)$ we have

$$\begin{aligned}f_{s-a+\delta}(H) \supseteq e_{s-a+\delta}(H) &= [M^\alpha[0, s-a+\delta]e(H)]\\ &\supseteq [M^\alpha[-b, -a]M^\alpha[t-2\delta, s+\delta]e(H)]\end{aligned}$$

(as $t - b - 2\delta > 0$). Using part (1) we now have

$$f_{s-a+\delta}(H) \supseteq [M^\alpha[-b, -a](f_s - f_t)(H)].$$

Hence $(I - f_{s-a+\delta})M^\alpha[-b, -a](f_s - f_t) = 0$. This implies (2) by taking adjoints and using the fact that

$$\vee\{I - f_{s-a+\delta} : \delta > 0\} = I - \wedge\{I - f_{s-a+\delta} : \delta > 0\} = I - f_{s-a}.$$

(3) For $b \geqq 0$ set in (2) $a = 0$, to get

$$(f_s - f_t)M^\alpha[0, b](I - f_s) = \{0\}.$$

As $s \to \infty$ we get $(f_\infty - f_t)M^\alpha[0, b](I - f_\infty) = \{0\}$ whenever $t > b$. Since $\wedge(f_t : t > b\} = f_b$ we have $(f_\infty - f_b)M^\alpha[0, b](I - f_\infty) = \{0\}$. ∎

LEMMA 3.4. *Let $\{f_t : t \in \mathbf{R}\}$, e and σ be as above.*

(1) *For every $t \geqq 0$, $(I - f_t)M^\alpha[0, t] \subseteq M^\sigma$.*

(2) $f_\infty M^\alpha[0, \infty)(I - f_\infty) \subseteq M^\sigma[0, \infty)$.

PROOF. (1) For $s > t \geqq b \geqq \varepsilon > 0$,

$$\begin{aligned}(f_{s+\varepsilon} - f_s)&M^\alpha[b - \varepsilon, b] \\ &= (f_{s+\varepsilon} - f_s)M^\alpha[b - \varepsilon, b](I - f_{s-b}) \quad \text{(Lemma 3.2(4))} \\ &= (f_{s+\varepsilon} - f_s)M^\alpha[b - \varepsilon, b](f_{s-b+2\varepsilon} - f_{s-b}) \quad \text{(Lemma 3.3(2))}.\end{aligned}$$

Hence, if x lies in $(f_{s+\varepsilon} - f_s)M^\alpha[b - \varepsilon, b]$, then (Lemma 2.7)

$$\begin{aligned}\mathrm{sp}_\sigma(x) &= \mathrm{sp}_\sigma(Q(s, s + \varepsilon]xQ(s - b, s - b + 2\varepsilon]) \\ &\subseteq [-s - \varepsilon, -s] + [b - \varepsilon, b] + [s - b, s - b + 2\varepsilon] \\ &\subseteq [-2\varepsilon, 2\varepsilon].\end{aligned}$$

We have $(f_{s+\varepsilon} - f_s)M^\alpha[b - \varepsilon, b] \subseteq M^\sigma[-2\varepsilon, 2\varepsilon]$.

Since $\bigcup\{M^\alpha[b - \varepsilon, b] : t \geqq b \geqq \varepsilon\}$ is σ-weakly dense in $M^\alpha[0, t]$, we have

$$(f_{s+\varepsilon} - f_s)M^\alpha[0, t] \subseteq M^\sigma[-2\varepsilon, 2\varepsilon] \qquad \text{for every } s > t \geqq \varepsilon > 0.$$

Since $\wedge\{f_s : s > t\} = f_t$ we have $\vee\{f_{s+\varepsilon} - f_s : s > t\} = f_\infty - f_t$. Hence

$$(f_\infty - f_t)M^\alpha[0, t] \subseteq M^\sigma[-2\varepsilon, 2\varepsilon] \qquad \text{for all } \varepsilon > 0.$$

Hence $(f_\infty - f_t)M^\alpha[0, t] \subseteq M^\sigma$ for $t > 0$. For $t = 0$ the assertion is trivial.

(2) For $x \in M^\alpha[0, \infty)$ we have

$$\begin{aligned}\mathrm{sp}_\sigma(f_0 x(I - f_\infty)) = \mathrm{sp}_\sigma(Q(0)f_\infty x(I - f_\infty)Q(1)) &\subseteq \{0\} + [0, \infty) + \{1\} \\ &\subseteq [0, \infty).\end{aligned}$$

Hence $f_0 M^\alpha[0, \infty)(I - f_\infty) \subseteq M^\sigma[0, \infty)$.

For $t \geqq 0$ and $\frac{1}{2} > \varepsilon > 0$ we have (Lemma 3.3(3))

$$(f_{t+\varepsilon} - f_t)M^\alpha[0, \infty)(I - f_\infty) = (f_{t+\varepsilon} - f_t)M^\alpha[t - \varepsilon, \infty)(I - f_\infty).$$

(We use the fact that $M^\alpha[0, \infty) = M^\alpha[0, t] + M^\alpha[t - \varepsilon, \infty)$; see Lemma 2.8.) Hence, for $x \in (f_{t+\varepsilon} - f_t)M^\alpha[0, \infty)(I - f_\infty)$,

$$\begin{aligned}\mathrm{sp}_\sigma(x) &= \mathrm{sp}_\sigma(Q(t, t+\varepsilon] f_\infty x Q(1))\\ &\subseteq [-t-\varepsilon, -t] + [t-\varepsilon, \infty) + \{1\}\\ &\subseteq [1-2\varepsilon, \infty)\\ &\subseteq [0, \infty).\end{aligned}$$

If follows that $(f_{t+\varepsilon} - f_t)M^\alpha[0, \infty)(I - f_\infty) \subseteq M^\sigma[0, \infty)$ for all $t \geqq 0$ and $\varepsilon > 0$. Since $f_\infty - f_0 = \vee\{f_{t+\varepsilon} - f_t : t \geqq 0\}$ for every $\varepsilon > 0$, we are done. ∎

LEMMA 3.5. *For e, $\{f_t : t \in \mathbf{R}\}$ and σ as above, $M^\alpha[0, \infty) \subseteq M^\sigma[0, \infty)$.*

PROOF. The proof that $M^\alpha[0, \infty) f_\infty \subseteq M^\sigma[0, \infty)$ proceeds almost precisely as in Proposition 2.10, using Lemma 3.4(1) instead of Corollary 2.4(3) and Proposition 2.6.

The fact that $f_\infty M^\alpha[0, \infty)(I - f_\infty) \subseteq M^\sigma[0, \infty)$ was proved in Lemma 3.4(2). It is left to prove

$$(I - f_\infty)M^\alpha[0, \infty)(I - f_\infty) \subseteq M^\sigma[0, \infty).$$

But $I - f_\infty = Q(1)(I - f_\infty)$. Hence, for $x \in M^\alpha[0, \infty)$,

$$\begin{aligned}\mathrm{sp}_\sigma((I - f_\infty)x(I - f_\infty)) &= \mathrm{sp}_\sigma(Q(1)(I - f_\infty)x(I - f_\infty)Q(1))\\ &\subseteq \{-1\} + [0, \infty) + \{1\}\\ &\subseteq [0, \infty).\end{aligned}$$

This completes the proof. ∎

We have thus shown that, given a non-zero projection e in $Z(M^\alpha)$, we can construct an action σ (of $\mathbf{R}$ on M) and the algebra $M^\sigma[0, \infty)$ contains $H^\infty(\alpha)$. We also know that for every $t \geqq 0$, $(1 - f_t)M^\alpha[0, t]$ is contained in M^σ where f_t is the projection onto $\bigcap\{[M^\alpha[0, s]e(H)] : s > t\}$. We shall write $B(e)$ for the algebra $M^\sigma[0, \infty)$.

LEMMA 3.6. *For every non-zero projection $e \in Z(M^\alpha)$ let $B(e)$ be the algebra defined above and suppose that for every such e, $B(e) = H^\infty(\alpha)$. If e is a projection in $Z(M^\alpha)$ satisfying $M^\alpha(a, b)e \neq 0$ (where $a < b$) and $\varepsilon > 0$, then $eM^\alpha(a - \varepsilon, b + \varepsilon)e \neq 0$.*

PROOF. For every non-zero projection $e \in Z(M^\alpha)$ we have constructed $\{f_t : t \in \mathbf{R}\}$ and they satisfy $(I - f_t)M^\alpha[0, t] \subseteq M^\sigma$. But, by assumption, $M^\sigma[0, \infty) = H^\infty(\alpha)$ and, therefore, $M^\sigma = M^\alpha$. We have, then, $(I - f_t)M^\alpha(0, t] = \{0\}$. Write

$$c_t = \sup\{\mathrm{rp}(y) : y \in M^\alpha(0, t]\}.$$

Then $c_t \in Z(M^\alpha)$ and $c_t \leqq f_t$ for every $t > 0$ (and every $e \neq 0$). Hence, for every non-zero projection $e \in Z(M^\alpha)$, every $s > t > 0$, and every non-zero projection $p \leqq c_t$ we have $pM^\alpha[0, s]e \neq \{0\}$ (since f_t is the projection onto $\bigcap \{[M^\alpha[0, s]e(H)] : s > t\}$).

Therefore $eM^\alpha[-s, 0]p \neq \{0\}$ for all such p and s and all non-zero projections e in $Z(M^\alpha)$. But this implies that for every $s > t > 0$ and $0 \neq p \leqq c_t$, $[M^\alpha[-s, 0]p(H)] = H$.

Now fix a non-zero projection e in $Z(M^\alpha)$ and an open interval $J = (a, b)$ in $\mathbf{R}$ satisfying $M^\alpha(a, b)e \neq \{0\}$ and set $\varepsilon > 0$. Write $e(J)$ for the projection onto $[M^\alpha(J)e(H)]$. Since $e(J) \neq 0$ (in $Z(M^\alpha)$) we have, for all $\varepsilon > t > 0$.

$$c_t M^\alpha[0, \varepsilon]e(J) \neq 0.$$

Let $r(J)$ be the projection onto $[c_t M^\alpha[0, \varepsilon]e(J)(H)]$. Then $0 \neq r(J) \leqq c_t$. Hence

$$[M^\alpha[-\varepsilon, 0]r(J)] = I$$

and, consequently,

$$eM^\alpha[-\varepsilon, 0)]c_t M^\alpha[0, \varepsilon]M^\alpha(a, b)e \neq 0.$$

Therefore $eM^\alpha(a - \varepsilon, b + \varepsilon)e \neq 0$. ■

We now turn to the main result of this section.

THEOREM 3.7. *Suppose $Z(M) \cap M^\alpha = \mathbf{C}I$. Then $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M if and only if either*

(i) $\mathrm{sp}(\alpha) = \Gamma(\alpha)$;

or

(ii) *there is a projection $F \in M$ such that*

$$H^\infty(\alpha) = \{x \in M : (I - F)xF = 0\}.$$

PROOF. We already know that (i) is a sufficient condition for maximality. If (ii) holds, then every σ-weakly closed subalgebra of M that contains $H^\infty(\alpha)$ is a nest subalgebra associated with a nest $\mathfrak{n} \subseteq \{0, F, I\}$ (see [3]). Hence $H^\infty(\alpha)$ is maximal.

Now assume that $H^\infty(\alpha)$ is maximal. For every non-zero projection $e \in Z(M^\alpha)$ we can construct projections $\{f_t, g_t : t \in \mathbf{R}\}$ and an action σ as in the discussion following Proposition 3.1. We write $B(e)$ for the algebra $M^\sigma[0, \infty)$.

Since $B(e) \supseteq H^\infty(\alpha)$ (Lemma 3.5) and $H^\infty(\alpha)$ is maximal, either $B(e) = H^\infty(\alpha)$ or $B(e) = M$.

Suppose that for some non-zero projection e in $Z(M^\alpha)$, $B(e) = M$. Then $M^\sigma = B(e) \cap B(e)^* = M$ (for the action σ associated with e). Thus $\sigma_t = \text{id}$ for all $t \in \mathbf{R}$. Since $\sigma_t(x) = U_t^* \alpha_t(x) U_t$, $t \in \mathbf{R}$, $x \in M$, we see that α_t is inner for every $t \in \mathbf{R}$. That implies [4] that $H^\infty(\alpha)$ is a nest subalgebra; i.e.

$$H^\infty(\alpha) = \{x \in M : (I - N)xN = 0 \text{ for every } N \in \mathfrak{N}\}$$

for some nest $\mathfrak{N}$ of projections in M. If $\mathfrak{N} = \{0, I\}$ we are done (take $F = 0$ in (ii)). Otherwise there is a projection $F \in \mathfrak{N}$ with $F \neq 0$, $F \neq I$. Then $H^\infty(\alpha) \subseteq \{x \in M : (I - F)xF = 0\}$. The algebra on the left is different from M since M is a factor (this follows from the condition $Z(M) \cap M^\alpha = \mathbf{C}I$, when α is inner). Therefore

$$H^\infty(\alpha) = \{x \in M : (I - F)xF = 0\}$$

and we are done.

Suppose now that there is no projection $e \neq 0$ in $Z(M^\alpha)$ such that $B(e) = M$; i.e. $B(e) = H^\infty(\alpha)$ for every non-zero projection $e \in Z(M^\alpha)$. We shall show that $\Gamma(\alpha) = \text{sp}(\alpha)$.

Fix t in $\text{sp}(\alpha)$ and $\delta > 0$. Define

$$N = \sup\{e \in Z(M^\alpha) : e \text{ is a projection and } M^\alpha(t - \delta, t + \delta)e = 0\}.$$

Then N is a projection in $Z(M^\alpha)$ and $M^\alpha(t - \delta, t + \delta)N = 0$. Now fix a non-zero projection e in $Z(M^\alpha)$. Since $t \in \text{sp}(\alpha)$, $N \neq I$. For every integer n define F_n to be the projection onto $[M^\alpha[n\delta, (n + 2)\delta]e(H)]$. Since $Z(M) \cap M^\alpha = \mathbf{C}I$, $[Me(H)] = H$. Hence $\bigvee\{F_n : n \in \mathbf{Z}\} = I$ (as the subspace spanned by $\bigcup\{M^\alpha[n\delta, (n + 2)\delta] : n \in \mathbf{Z}\}$ is σ-weakly dense in M). There is, therefore, some $n \in \mathbf{Z}$ with $F_n \nleq N$; i.e. $M^\alpha(t - \delta, t + \delta)F_n \neq 0$. Now apply Lemma 3.6 to conclude that $F_n M^\alpha(t - \delta, t + \delta)F_n \neq \{0\}$. Hence, for some ζ and η in H and x in $M^\alpha(t - \delta, t + \delta)$,

$$\langle xF_n\zeta, F_n\eta \rangle \neq 0.$$

But we can assume tht $F_n\eta = ye\eta'$ for some $y \in M^\alpha[n\delta, (n + 2)\delta]$ and $\eta' \in H$. Hence $\langle ey^*xF_n\zeta, \eta' \rangle \neq 0$. This implies that

$$eM^\alpha[-(n + 2)\delta, -n\delta]M^\alpha(t - \delta, t + \delta)F_n(H) \neq \{0\}.$$

Hence,

$$eM^\alpha(t-3\delta, t+3\delta)e$$
$$\supseteq eM^\alpha[-(n+2)\delta, -n\delta]M^\alpha(t-\delta, t+\delta)M^\alpha(n\delta, (n+2)\delta)e \neq \{0\}.$$

Thus, for every $\delta > 0$, $eM^\alpha(t-3\delta, t+3\delta)e \neq 0$; hence $t \in \operatorname{sp}(\alpha^e)$. Since this holds for every non-zero projection e in $Z(M^\alpha)$, $t \in \Gamma(\alpha)$. ∎

References

1. W. B. Arveson, *On groups of automorphisms of operator algebras*, J. Funct. Anal. **15** (1974), 217–243.
2. S. Kawamura and J. Tomiyama, *On subdiagonal algebras associated with flows in operator algebras*, J. Math. Soc. Japan **29** (1977), 73–90.
3. D. R. Larson and B. Solel, *Nests and inner flows*, J. Operator Theory **16** (1986), 157–164.
4. R. I. Loebl and P. S. Muhly, *Analyticity and flows in von Neumann algebras*, J. Funct. Anal. **29** (1978), 214–252.
5. A. I. Loginov and V. S. Sulman, *The hereditary and intermediate reflexivity for W*-algebras*, Izv. Akad. Nauk SSSR **39** (1975), 1260–1273.
6. M. McAsey, P. S. Muhly and K.-S. Saito, *Nonselfadjoint crossed products (Invariant subspaces and maximality)*, Trans. Amer. Math. Soc. **248** (1979), 381–409.
7. M. McAsey, P. S. Muhly and K.-S. Saito, *Nonselfadjoint crossed products II*, J. Math. Soc. Japan **33** (1981), 485–495.
8. M. McAsey, P. S. Muhly and K.-S. Saito, *Nonselfadjoint crossed products III (Infinite algebras)*, J. Operator Theory **12** (1984), 3–22.
9. P. S. Muhly, *Function algebras and flows*, Acta Sci. Math. (Szeged) **35** (1973), 111–121.
10. P. S. Muhly and K.-S. Saito, *Analytic subalgebras of von Neumann algebras*, preprint.
11. P. S. Muhly, K.-S. Saito and B. Solel, *Coordinates for triangular operator algebras*, preprint.
12. W. Rudin, *Fourier analysis on groups*, Interscience Publishers, New York–London, 1967.
13. B. Solel, *Nonselfadjoint crossed products: Invariant subspaces, cocycles and subalgebras*, Indiana Univ. Math. J. **34** (1985), 277–298.
14. B. Solel, *Algebras of analytic operators associated with a periodic flow on a von Neumann algebra*, Can. J. Math. **37** (1985), 405–429.
15. S. Stratila, *Modular theory in operator algebras*, Abacus Press, Tunbridge, England, 1981.
16. M. Takesaki, *Theory of operator algebras I*, Springer-Verlag, New York–Heidelberg–Berlin, 1979.